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LETTER TO THE EDITOR

Computer-friendly d-tensor identities for SU(n)

A Sudbery†

Department of Mathematics, University of York, Heslington, York YO1 5DD, UK

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Abstract. The identity of degree $n-1$ satisfied by the SU(n) tensor d_{ijk} is derived and presented in a simple recursive form, suitable for computation.

The algebra of SU(n) tensors has recently become of interest in the study of conformal field theories, following the construction by Bais, Bouwknegt, Schoutens and Surridge [1] of bosonic extensions of the Virasoro algebra using higher-order Casimir invariants of Lie algebras. For the A_n series these invariants can be formed by means of the tensor d_{ijk}, which satisfies a number of identities. Some of these take the same form for all n, but there is one identity whose form depends on n. A general formulation of this identity has been given by Rashid and Saifuddin [2], but this involves a sum over partitions of n and it is laborious to use it to obtain the specific form of the identity for any particular value of n. The purpose of this letter is to present an independent derivation of this identity, leading to a form involving tensors defined by simple recursive formulae, which is suitable for use in computation, whether by hand or by machine.

We use the same notation and conventions as in [3]. Indices i, j, k run from 1 to $N=n^2-1$, labelling coordinates in the adjoint representation of su(n) ($=A_{n-1}$). The tensors f_{ijk} and d_{ijk}, totally antisymmetric and symmetric respectively, are defined by the multiplication rules for a basis set V_i of $n\times n$ Hermitian matrices:

$$V_iV_j=\frac{2}{n}\,\delta_{ij}+(d_{ijk}+\mathrm{i}f_{ijk})\,V_k. \tag{1}$$

They satisfy the following identities [3] for all n:

$$f_{imn}f_{njk}+f_{jmn}f_{ink}+f_{kmn}f_{ijn}=0 \tag{2}$$

$$f_{imn}d_{njk}+f_{jmn}d_{ink}+f_{kmn}d_{ijn}=0 \tag{3}$$

$$f_{ijk}f_{mnk}=\frac{2}{n}\,(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})+(d_{imk}d_{jnk}-d_{ink}d_{jmk}) \tag{4}$$

$$f_{imn}f_{jmn}=n\delta_{ij} \tag{5}$$

$$d_{imn}d_{jmn}=\left(\frac{n^2-4}{n}\right)\delta_{ij}. \tag{6}$$

The further identity which is the subject of this letter is a consequence of the Cayley–Hamilton theorem. This identity, whose form is specific to the value of n, is

† E-mail: AS2@VAXA.YORK.AC.UK

a tensor equation of rank $n+1$ involving δ_{ij} and d_{ijk} only. It is best discussed in terms of the rth-rank tensors $d^{(r)}_{i_1\ldots i_r}$ $(r\geq 2)$ formed from repeated products of d_{ijk}:

$$d^{(r+1)}_{i_1\ldots i_{r+1}} = d^{(r)}_{i_1\ldots i_{r-1}j}d_{ji_ri_{r+1}} \qquad \text{with} \qquad d^{(2)}_{ij} = \delta_{ij} \tag{7}$$

or

$$d^{(r)}_{i_1\ldots i_r} = (D_{i_2}\ldots D_{i_{r-1}})_{i_1 i_r} \tag{8}$$

where D_i is the $N\times N$ matrix whose (j, k)th element is d_{ijk}.

It is convenient to make the set of matrices V_i into a complete set of $n\times n$ matrices by annexing the unit matrix: $V_0=\sqrt{(2/n)}\,1$. In analogy with the established convention for Minkowski space, we will use Latin indices to run from 1 to N and Greek ones to run from 0 to N. We now have

$$\mathrm{Tr}(V_\mu V_\nu) = 2\delta_{\mu\nu} \tag{9}$$

and

$$V_\mu V_\nu = e_{\mu\nu\lambda}V_\lambda + \mathrm{i}f_{\mu\nu\lambda}V_\lambda \tag{10}$$

where $e_{0\mu\nu}=\sqrt{(2/n)}\,\delta_{\mu\nu}$, $e_{ijk}=d_{ijk}$ and $f_{0\mu\nu}=0$. (Note that we do not spoil the covariance of a tensor by putting one or more of its indices equal to 0; this is because the space of the V_μ is reducible under the SU(n) transformations $V\to UVU^{-1}$.)

As before, we define the rth-rank tensors $e^{(r)}_{\mu_1\ldots\mu_r}$ by

$$e^{(r+1)}_{\mu_1\ldots\mu_{r+1}} = e^{(r)}_{\mu_1\ldots\mu_{r-1}\nu}e_{\nu\mu_r\mu_{r+1}}. \tag{11}$$

This series starts with a tensor of rank one: $e^{(1)}_\mu=\sqrt{(2/n)}\,\delta_{\mu 0}$. When one of its indices is 0, $e^{(r+1)}$ reduces to $e^{(r)}$:

$$e^{(r+1)}_{\mu_1\ldots 0\ldots\mu_r} = \sqrt{\frac{2}{n}}\,e^{(r)}_{\mu_1\ldots\mu_r} \tag{12}$$

and when none of its indices is 0, an e-tensor can be expressed in terms of d-tensors by a recursion formula:

$$e^{(r)}_{i_1\ldots i_r} = \frac{2}{n}\sum_{s=0}^{r-2} e^{(s)}_{i_1\ldots i_s}d^{(r-s)}_{i_{s+1}\ldots i_r} \tag{13}$$

interpreting $e^{(0)}=n/2$.

The usefulness of e-tensors lies in the following expression for a power of the matrix $A=a_\mu V_\mu$:

$$(a_\mu V_\mu)^r = e^{(r+1)}_{\mu_1\ldots\mu_r\nu}a_{\mu_1}\ldots a_{\mu_r}V_\nu. \tag{14}$$

Let us write the characteristic polynomial of a matrix A as

$$P_A(x) = x^n - \Delta_1(A)x^{n-1} + \Delta_2(A)x^{n-2} - \ldots + (-)^n\Delta_n(A). \tag{15}$$

The coefficients $\Delta_r(A)$ are the invariants of the matrix A. $\Delta_r(A)$ is the sum of all products of r different eigenvalues of A (an m-fold degenerate eigenvalue counts as m different eigenvalues for our purposes). The invariants of A can be expressed in terms of traces of powers of A, independently of the dimension n, by Newton's formula

$$\Delta_r(A) = \frac{1}{r}\sum_{s=0}^{r-1}(-)^{r-s+1}\Delta_s(A)\,\mathrm{Tr}(A^{r-s}) \tag{16}$$

with $\Delta_0(A)=1$.

If we write $A=a_\mu V_\mu$, it is clear that $\Delta_r(A)$ is a homogeneous polynomial of degree r in a_μ:

$$\Delta_r(A)=\Delta^{(r)}_{\mu_1\ldots\mu_r}a_{\mu_1}\ldots a_{\mu_r}. \tag{17}$$

Also, from (12) and (14), we have

$$\mathrm{Tr}(A^r)=2e^{(r)}_{\mu_1\ldots\mu_r}a_{\mu_1}\ldots a_{\mu_r}. \tag{18}$$

Thus the tensors $\Delta^{(r)}$ can be defined recursively by

$$r!\Delta^{(r)}_{\mu_1\ldots\mu_r}=\frac{2}{r}\underset{\mu_1\ldots\mu_r}{\mathrm{S}}\sum_{s=0}^{r-1}(-)^{r-s+1}\Delta^{(s)}_{\mu_1\ldots\mu_s}e^{(r-s)}_{\mu_{s+1}\ldots\mu_r} \tag{19}$$

where the symbol S denotes complete symmetrisation of the indices over which it stands.

Now the Cayley-Hamilton theorem states that

$$P_A(A)=A^n-\Delta_1(A)A^{n-1}+\Delta_2(A)A^{n-2}-\ldots+(-)^n\Delta_n(A)=0. \tag{20}$$

Putting $A=a_\mu V_\mu$ and using (14) for the powers of A, we obtain

$$\sum_{s=0}^{n}(-)^s\Delta^{(s)}_{\mu_1\ldots\mu_s}e^{(n-s+1)}_{\mu_{s+1}\ldots\mu_n\nu}a_{\mu_1}\ldots a_{\mu_n}=0 \tag{21}$$

by virtue of the linear independence of the V_μ. Since this equation is true for arbitrary a_μ, we deduce that

$$\underset{\mu_1\ldots\mu_n}{\mathrm{S}}\sum_{s=0}^{n}(-)^s\Delta^{(s)}_{\mu_1\ldots\mu_s}e^{(n-s+1)}_{\mu_{s+1}\ldots\mu_n\nu}=0. \tag{22}$$

This contains no new information if one of the indices is 0. If $\nu=0$, it merely restates the definition of $\Delta^{(n)}$; if one of the μ_i is 0, it follows from the formula

$$\Delta^{(r)}_{\mu_1\ldots\mu_s0\ldots0}=\left(\frac{n}{2}\right)^{-\frac{1}{2}(r-s)}\frac{(n-s)!\,s!}{(n-r)!\,r!}\Delta^{(s)}_{\mu_1\ldots\mu_s}. \tag{23}$$

This results solely from the manner of construction of $\Delta^{(r)}$, irrespective of the nature of d_{ijk}, for it can be proved from (19) and (21) if one has the inductive stamina. More easily, it can be seen to be equivalent to

$$\Delta_r(A+\alpha 1)=\sum_{s=0}^{r}\binom{n-s}{r-s}\Delta_s(A)\alpha^{r-s} \tag{24}$$

which is a direct consequence of the interpretation of $\Delta_r(A)$ in terms of eigenvalues.

Thus the identity at which we have been aiming is

$$\underset{i_1\ldots i_n}{\mathrm{S}}\sum_{s=0}^{n}(-)^s\Delta^{(s)}_{i_1\ldots i_s}e^{(n-s+1)}_{i_{s+1}\ldots i_nj}=0 \tag{25}$$

where the tensors $e^{(r)}$ and $\Delta^{(r)}$ are defined recursively by (13) and (19). The main feature of this identity is that it expresses the tensor $d^{(n+1)}$, symmetrised with respect to all but the last of its indices, as a sum of uncontracted products of d-tensors of lower rank. To illustrate this, here are the forms taken by (2.19) for a few low values of n:

$$n=2: \qquad d_{ijk}=0 \tag{26}$$

$$n=3: \qquad d^{(4)}_{\underline{ijk}l}=\tfrac{1}{3}\,\delta_{\underline{ij}}\delta_{\underline{k}l} \tag{27}$$

$$n=4: \qquad d^{(5)}_{\underline{ijk}lm}=\tfrac{1}{2}\delta_{\underline{ij}}d_{\underline{k}lm}+\tfrac{1}{6}d_{\underline{ijk}}\delta_{\underline{l}m} \tag{28}$$

$$n=5: \qquad d^{(6)}_{\underline{ijklm}n}=\tfrac{3}{5}\delta_{\underline{ij}}d^{(4)}_{\underline{klm}n}+\tfrac{1}{10}d^{(4)}_{\underline{ijkl}}\delta_{\underline{m}n}+\tfrac{4}{15}d_{\underline{ijk}}d_{\underline{lm}n}-\tfrac{3}{50}\delta_{\underline{ij}}\delta_{\underline{kl}}\delta_{\underline{m}n} \tag{29}$$

(underlined indices are to be totally symmetrised).

SU(n) vector algebra. Ordinary three-dimensional vectors can be regarded as SU(2) tensors of rank one, i.e. elements of the Lie algebra A_1. Conventional vector algebra can be generalised to the algebra of SU(n) vectors. There is an antisymmetric vector product $\boldsymbol{a} \wedge \boldsymbol{b}$ deriving from the Lie bracket:

$$(\boldsymbol{a} \wedge \boldsymbol{b})_i=f_{ijk}a_jb_k. \tag{30}$$

There is also a symmetric vector product formed with d_{ijk}:

$$(\boldsymbol{a} * \boldsymbol{b})_i=d_{ijk}a_jb_k. \tag{31}$$

Neither of these vector products is associative.

Equations (2)-(4) give rise to the following connections between the two vector products:

$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c})+\boldsymbol{b}\wedge(\boldsymbol{c}\wedge\boldsymbol{a})+\boldsymbol{c}\wedge(\boldsymbol{a}\wedge\boldsymbol{b})=0 \tag{32}$$

$$\boldsymbol{a}\wedge(\boldsymbol{b}*\boldsymbol{c})+\boldsymbol{b}\wedge(\boldsymbol{c}*\boldsymbol{a})+\boldsymbol{c}\wedge(\boldsymbol{a}*\boldsymbol{b})=0 \tag{33}$$

$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c})=\frac{2}{n}[(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}]+[(\boldsymbol{a}*\boldsymbol{c})*\boldsymbol{b}-(\boldsymbol{a}*\boldsymbol{b})*\boldsymbol{c}]. \tag{34}$$

Starting from a given vector $\boldsymbol{a}$ and forming repeated $*$-products, we generate in general an $(n-1)$-dimensional space H_a in which the $\wedge$-product is null. (This is an Abelian subalgebra of the Lie algebra of SU(n), and since this group has rank $n-1$, the dimension of H_a cannot exceed $n-1$.) An expression $\boldsymbol{a}*\boldsymbol{a}*\ldots*\boldsymbol{a}$ is ambiguous, for its value depends on the order in which brackets are inserted into the product, but by using (4) any such product can be expressed as a combination of the following basis vectors for H_a:

$$\boldsymbol{a}^{*r}=\boldsymbol{a}*(\boldsymbol{a}*(\ldots*\boldsymbol{a})\ldots) \qquad (r \text{ factors}) \tag{35}$$

i.e.

$$(\boldsymbol{a}^{*r})_i=d^{(r+1)}_{ij_1\ldots j_r}a_{j_1}\ldots a_{j_r}. \tag{36}$$

In general, $\boldsymbol{a}, \boldsymbol{a}*\boldsymbol{a}, \ldots, \boldsymbol{a}^{*n-1}$ are linearly independent; the identity (25) enables $\boldsymbol{a}^{*n}$ to be expressed in terms of them. The following relations exist:

$$\boldsymbol{a}^{*r}\wedge\boldsymbol{a}^{*s}=0 \tag{37}$$

$$\boldsymbol{a}\cdot(\boldsymbol{a}^{*r-1}*\boldsymbol{a}^{*s})=\boldsymbol{a}^{*r}\cdot\boldsymbol{a}^{*s}=a^{r+s} \tag{38}$$

where the invariants a^p are defined by

$$a^p=d^{(p)}_{i_1\ldots i_p}a_{i_1}\ldots a_{i_p}. \tag{39}$$

These $n-1$ independent invariants correspond to the Casimir operators of SU(n).

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